FIR Filters

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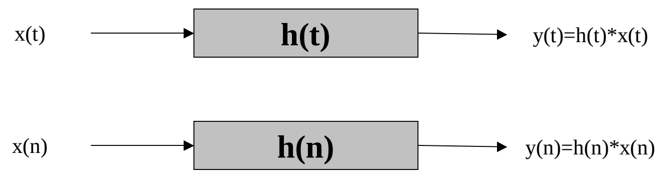
Filters

- Filters are used to change the input signal in some way
  - Filter out unwanted signals
  - Enhance a good signal
  - Adjust the phase of a signal
  - Detect signal into noise

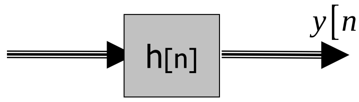
LTI systems



$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

The Math

$$x[n] = e^{j\omega n}$$



$$y[n] = x[n] * h[n] = e^{j\omega n} * h[n]$$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)} h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} e^{-j\omega k} h[k] \\ &= e^{j\omega n} H\left(e^{j\omega}\right) = e^{j\omega n} H(\omega) \end{aligned}$$

$$H\left(e^{j\omega}\right) \equiv \sum_{k=-\infty}^{\infty} e^{-j\omega k} h[k]$$

- H(ω) is the frequency response of the filter

Frequency Response

$$H(\omega) = H\left(e^{j\omega}\right) \equiv \sum_{k=-\infty}^{\infty} e^{-j\omega k} h[k]$$

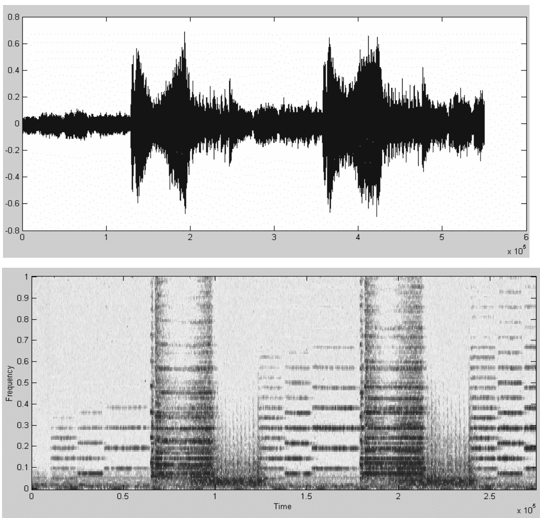
$$= H_{\text{Re}}\left(e^{j\omega}\right) + jH_{\text{Im}}\left(e^{j\omega}\right)$$

$$= \left|H\left(e^{j\omega}\right)\right| e^{j\, ph\left(H\left(e^{j\omega}\right)\right)}$$

- H(ω) is the <u>frequency response</u> of the filter-another way to represent a system (impulse response)
  - Magnitude is called the frequency spectrum
  - Both magnitude and phase important in filter design

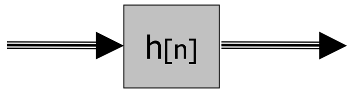
Frequency Spectrum

- Shows what frequencies are present in a signal
- Time signal shows sound intensity and some frequency
- Spectrogram shows which signals are present at a given time.



Eigenfunctions of a System

$x[n] = e^{j\omega n}$



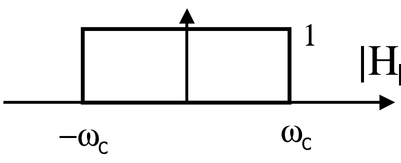
$y[n] \quad = e^{j\omega n} * h[n]$

$$y[n] \quad = e^{j\omega n} H\left(e^{j\omega}\right) = e^{j\omega n} H(\omega) \qquad H\left(e^{j\omega}\right) \equiv \sum_{k=-\infty}^{\infty} e^{-j\omega k} h[k]$$

- Note: output has the same frequency as the input signal
- Special characteristic of an LTI system: sinusoid in gives an output at the <u>same</u> frequency => NO NEW FREQUENCIES
- LTI system changes phase and amplitude not frequency

Ideal Filters

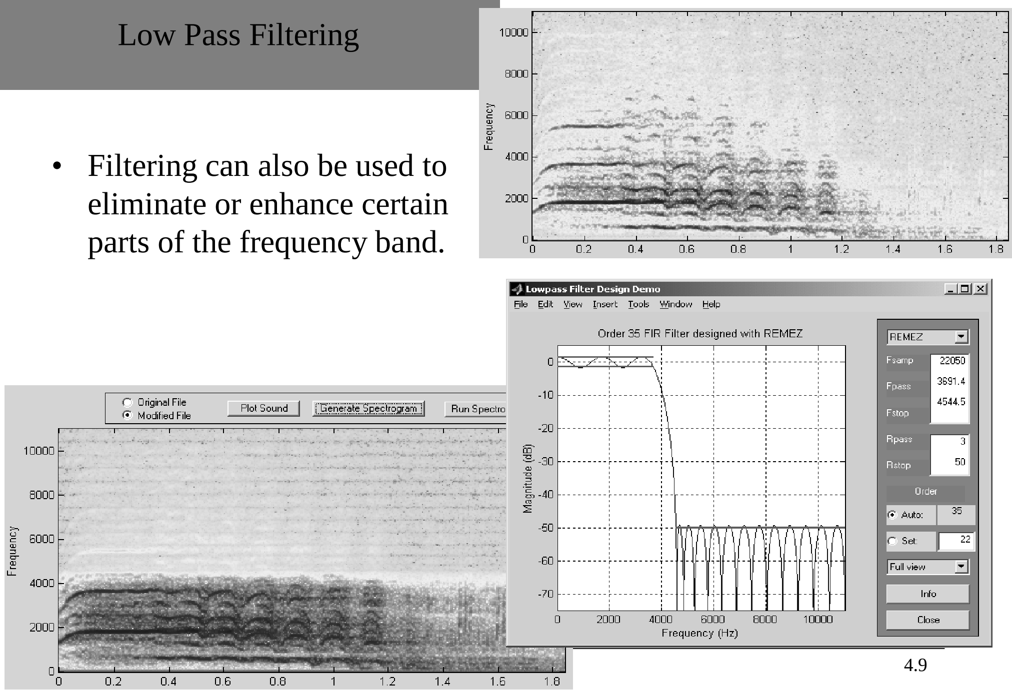
- Low-Pass Filters: Allow low frequencies to pass, blurs the signal



$|H_{lp}(\omega_c)| = 1 \ \text{ if } |\omega| < \omega_c$

Low Pass Filtering

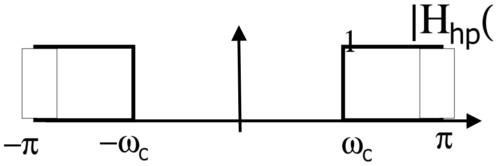
- Filtering can also be used to eliminate or enhance certain parts of the frequency band.



Other Ideal Filters

- High-Pass Filters: Allows high frequencies to pass, sharpens edges and fine details

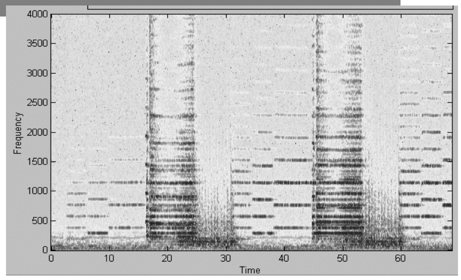
$$|H_{hp}(\omega_c)|=1 \quad \text{if } |\omega| > \omega_c$$

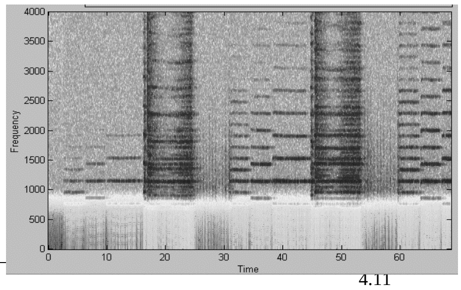


- BandPass Filters: Allows selected frequencies to pass

High Pass Filtering Example

- Keeps all high sounds
- Eliminates low sounds



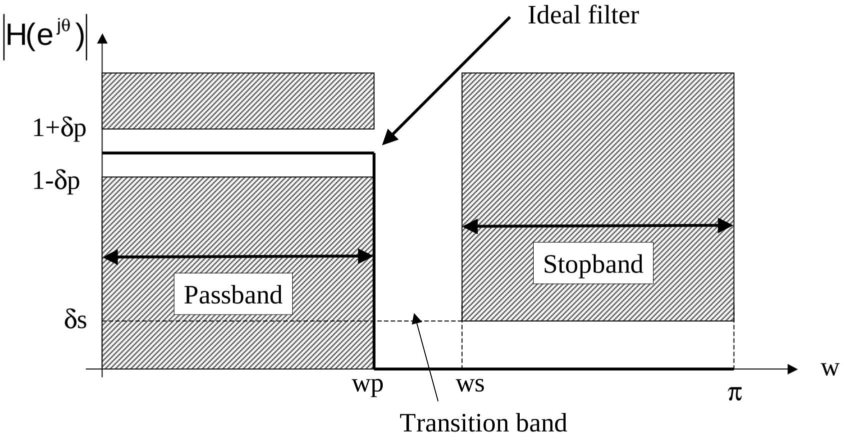


Notes on Ideal Filters

- These filters cannot be built in real-life due to the time-frequency duality problem
  - If a signal is time-limited, then it must be infinite in frequency
  - If a signal is frequency-limited, then it must be infinite in time

FIR filters

Specification from ideal filters



The Discrete-Time Fourier Transform (DTFT)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{jn\omega}d\omega \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}$$

- The first equation asserts that we can represent any time function *x[n]* by a linear combination of complex exponentials

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

- The second equation tells us how to compute the complex weighting factors $X(e^{j\omega})$

- In going from the DTFT to the ZT we replace $e^{j\omega n}$ by z^n

FIR filters

FIR Filters:

$$H(e^{j\theta}) = \sum_{n=0}^{N-1} h(n)\ e^{-jn\theta} \qquad H(z) = \sum_{n=0}^{N-1} h(n)\ z^{-n}$$

We know $H(e^{j\theta})$

How do we find h(n)?

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{jn\omega}d\omega$$

FIR filters

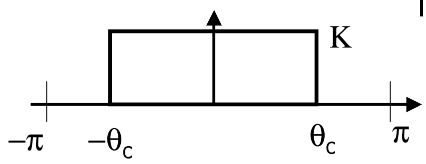
How to find h(n) for sampled signals ?

$$H(e^{jwT_s}) = \sum_{n=0}^{N-1} h(nT_s)\ e^{-jnwT_s}$$

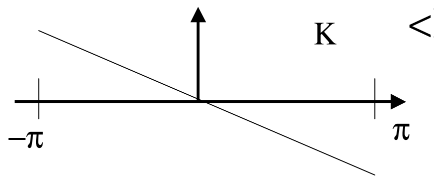
$$h[nT_s] = \frac{1}{w_s}\int_{-\frac{w_s}{2}}^{\frac{w_s}{2}} H(e^{jwT_s})e^{jnwT_s}dw \qquad w_s = 2\pi f_s$$

Low Pass design

$|H(\theta_c)| = K \text{ if } |\omega| < \omega_c$



$\angle H(\theta_c)$



Low Pass design

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})e^{jn\theta}d\theta$$

$$= \frac{1}{2\pi}\int_{-\theta_c}^{\theta_c} Ke^{jn\theta}d\theta = \frac{K}{2\pi}\left[\frac{e^{jn\theta}}{jn}\right]_{-\theta_c}^{\theta_c}$$

$$= \frac{K}{n\pi}\sin(n\theta_c)$$

Infinite number of coefficients, non causal!

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Low Pass design

Solution: Take only a first N coefficients

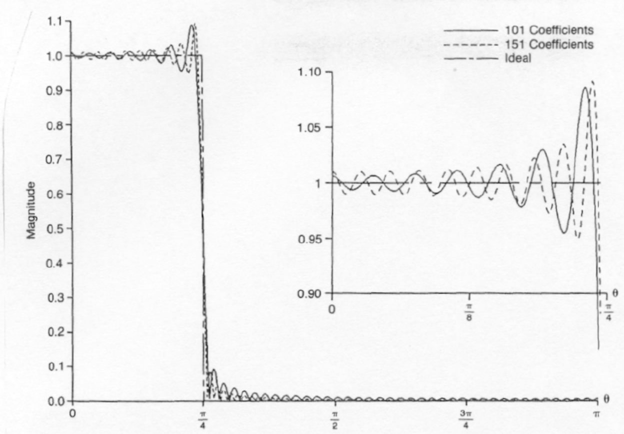
$$y(n) = \sum_{k=-N}^{N} h(k)x(n-k)$$

Still non-causal!

Shift them using a linear phase term!

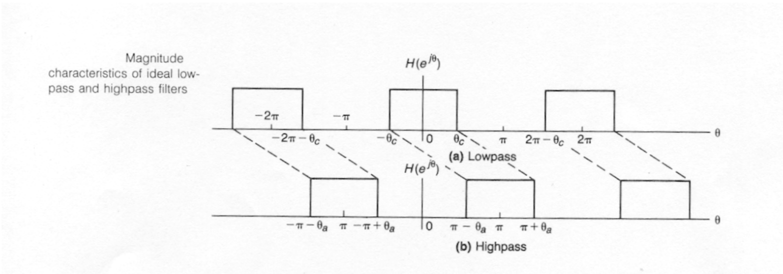
$$H(e^{j\theta}) = Ke^{-jN\theta}$$

Low Pass design



Lowpass filter frequency response for 101- and 151-coefficient designs

Low Pass / High Pass Filter



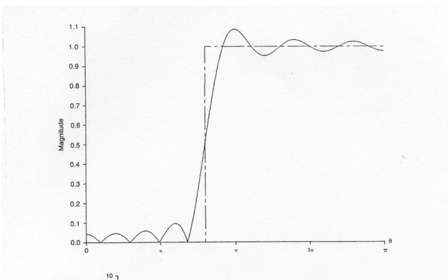
$$H_{hp}(e^{j\theta}) = \sum_{n=0}^{N-1} h_{hp}(n)\ e^{-jn\theta} = H_{lp}(e^{j(\theta-\pi)})$$

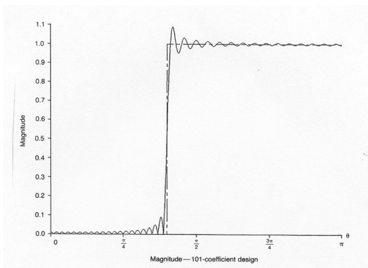
$$= \sum_{n=0}^{N-1} h_{lp}(n)\ e^{-jn(\theta-\pi)} = \sum_{n=0}^{N-1} h_{lp}(n)\ e^{-jn\theta} e^{jn\pi}$$

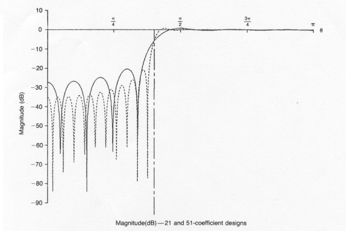
$$= \sum_{n=0}^{N-1} h_{lp}(n)\ (-1)^n e^{-jn\theta}$$

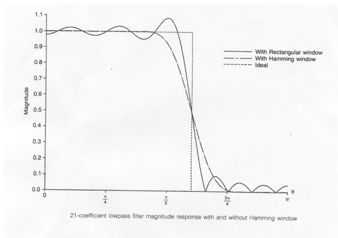
$$h_{hp}(n) = h_{lp}(n)\ (-1)^n$$

High Pass Filter

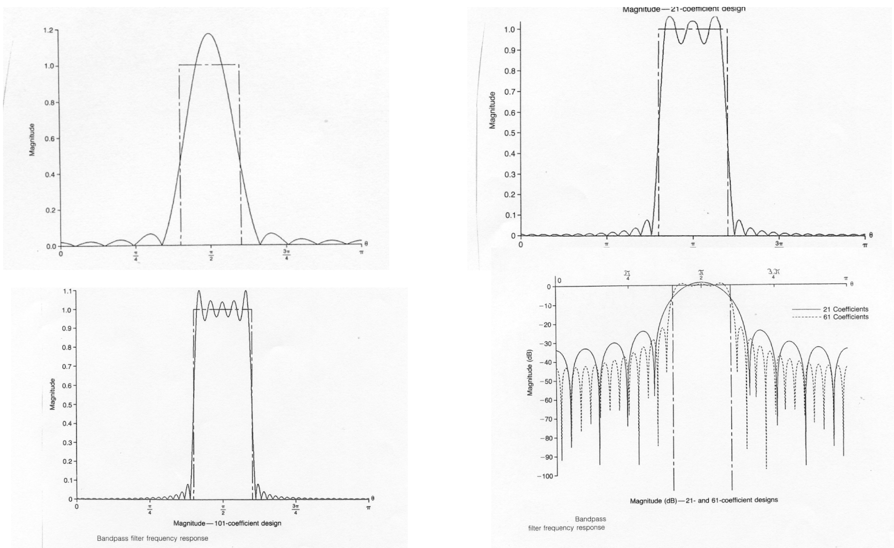








Band Pass design

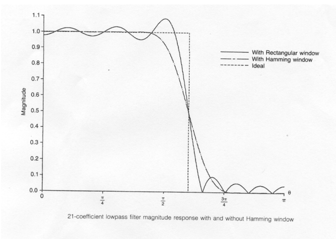


Gibbs Phenomenon

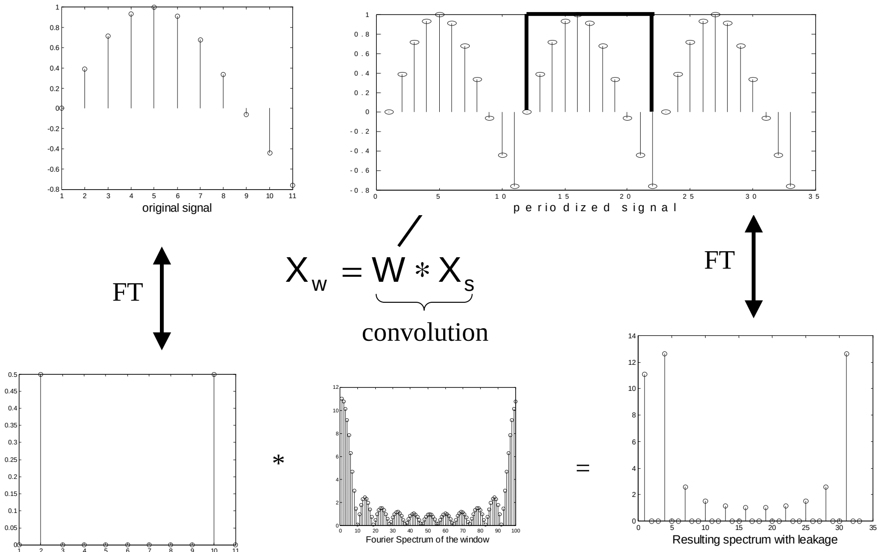
Finite number of coefficients:

$\hat{h}[n] = h(n).w(n)$ $\quad$ $w(n)$ windowing function

Same problem as for spectrum analysis!

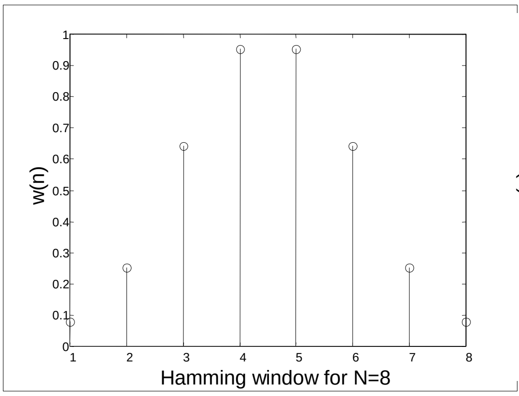


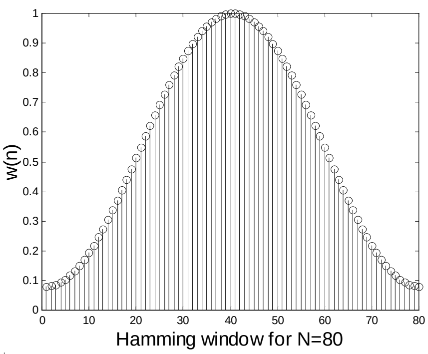
Square window



Optimised classical windows

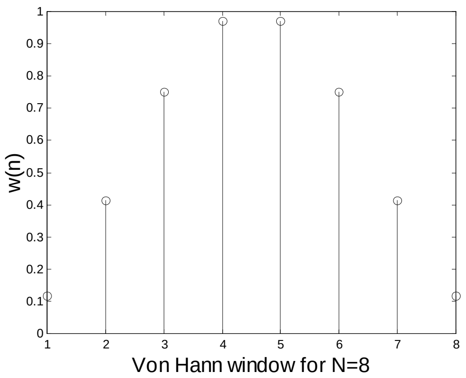
Hamming: $w(n) = 0.54 - 0.46 * \cos\left(\frac{2\pi n}{N-1}\right)$

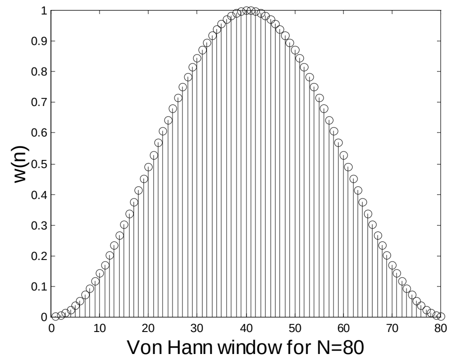




Optimised classical windows

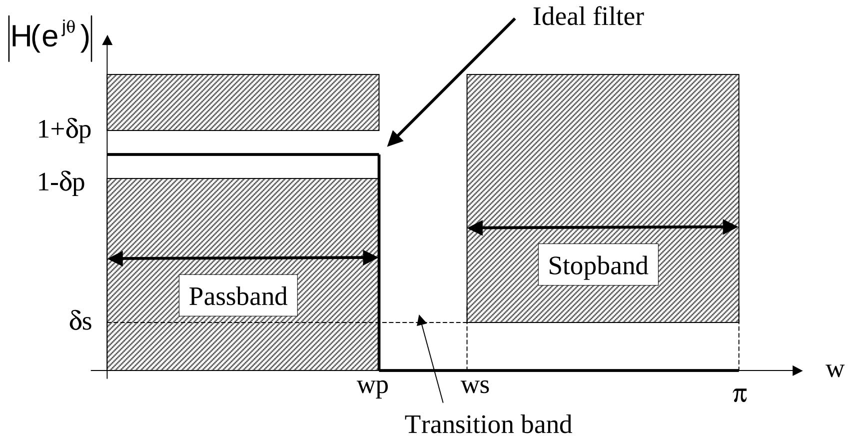
Hanning: $w(n) = 0.50 - 0.50 * \cos\left(\frac{2\pi n}{N-1}\right)$





FIR filters

Specification from ideal filters



Optimised classical windows

Difference between windows ?

Window type	Mathematical expression	Sidelobes (dB)	Transition width	Stop band attenuation
Rectangular	$w(n) = 1, \quad 0 \le n \le N-1$	-13	0.9/N	-21
Hamming	$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n \le N-1$	-31	3.1/N	-44
Hanning	$w(n) = 0.50 - 0.40\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n \le N-1$	-41	3.3/N	-53
Blackman	$w(n) = 0.42 - 0.50\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right) \quad 0 \le n \le N-1$	-57	5.5/N	-74