Decomposition of the OTF: Wavefront Coding as a Cornu Spiral

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Summary

A novel insight into the fundamental principles of wavefront coding is described; we show that the optical transfer function can be considered in terms of a generalised Cornu spiral and consequently important performance parameters can be derived in a greatly simplified and informative manner from the geometry of such a spiral.

Introduction

Recent years have witnessed the development of a new imaging technique, known as wavefront coding, that employs an aspherical phase mask and post-detection electronic processing to enable high-performance imaging with far greater tolerance to defocus aberration than traditional optical imaging [1]. Various methods have been employed for synthesizing the design of wavefront-coding phase mask [1,2]; a frequency domain analysis using the ambiguity function indicated that the ideal mask for an extended depth of field has a linearly separable cubic form. Although these techniques are efficient in synthesizing the shape of wavefront coding masks against specific criteria, the performance parameters are not amenable to simple calculation and interpretation and nor do they readily yield a physical insight into the underlying process.

We describe a new insight into the fundamental physical principles of wavefront coding and show how its functioning can be reduced to a simple geometrical depiction of a spiral curve from which to obtain the performance parameters.

OTF as a composition of Young's fringes

For the sake of clarity and without loss of generality, the analysis is restricted to the case of a one-dimensional optical system. It is well known that the optical transfer function (OTF) of an imaging system for spatially incoherent illumination is determined by the normalised autocorrelation of the pupil function P(x) of unit half length [3]

$$L(\nu) = \int_{-1}^{1} P(x+\nu)\, P^*(x-\nu)\, dx \Big/ \int_{-1}^{1} |P(x)|^2\, dx \qquad (1)$$

where x is the normalised transverse linear co-ordinate and ν is the normalised spatial frequency. Physically, Eq. (1) can be interpreted in terms of the integral of Young's fringes of spatial frequency ν due to the integral of all possible aperture pairs of separation 2ν within the pupil. It will be seen that decomposing the integral in Eq. (1) into the complex amplitudes of the component interferograms helps us to understand the composition of the OTF L(ν). To this end we define the integral

$$H(x',\nu) = \int_{0}^{x'} h(x,\nu)\, dx \Big/ \int_{-1}^{1} h(x,0)\, dx \qquad (2)$$

with $h(x,\nu)=P(x+\nu)\cdot P^*(x-\nu)$ and $h(x,0)=|P(x)|^2$. We note that the normalised OTF is $L(\nu)=H(1-\nu, \nu) - H(-1+\nu, \nu)$. This enables the contributions to the OTF to be depicted in a manner redolent to how the Cornu spiral describes diffraction of coherent light at an edge; the caveat is that the phasors h(x, ν), represent interference fringes rather than the optical field. We represent incremental contributions hdx by phasors and perform the integral in Eq. (2) as x' is varied between zero and the negative limit $-1+\nu$ and between zero and the positive limit $1-\nu$.

In the simple case of a clear pupil P(x)=1, the phasor h(x, ν) is real for all x and ν, the OTF L(ν) is then purely real and L(ν)=$1-\nu$ has the characteristic triangular form due to the linear variation of the autocorrelation function with ν.

We consider now a defocused optical system in which a cubic phase mask is inserted in the exit pupil. The pupil function for this system is $P(x)=\mathrm{Exp}[i2\pi(w_{20}x^2+\alpha x^3)]$, where w_{20} is the defocus coefficient and α is the strength of the phase mask, both in units of wavelength. The interferogram phasors are then given by $h(x,\nu;w_{20},\alpha)=\mathrm{Exp}[i4\pi(3\alpha x^2+2w_{20}x+\alpha\nu^2)]$. The complex decomposition of the OTF is obtained by inserting the phasor expression $h(x,\nu;w_{20},\alpha)$ into Eq. (2) and plotting the variation of $H(x',\nu;w_{20},\alpha)$ as x' varies between $-1+\nu$ and $1-\nu$.

The resulting integral in Eq. (2) describes arc-length parameterised curves known as generalised Cornu spirals (GCS) or clothoids [4]. $H(x',\nu;w_{20},\alpha)$ is plotted in Fig.1(a) for α=2, ν=0.5 and various w_{20}. Also are shown the corresponding MTFs for this system. The resultant OTF at frequency ν, is indicated by the phasor between the inner ends of the two spirals. It can be seen that for increasing defocus one spiral becomes more tightly wound whilst the opposite spiral unwinds in a manner such that, in terms of the MTF, the effect of one approximately counteracts the other. In consequence, the MTF remains approximately constant until one spiral is completely unwound. Physically, the coiling for positive and negative x' represents the resultant of the combined effects of the cubic phase mask and the defocus; a tightening in the spirals occurs when the cubic phase mask is in the same sense as the quadratic defocus and loosening of the spirals occurs when the cubic phase mask tends to cancel the quadratic defocus. For a conventional lens without wavefront coding α=0, $H(x',\nu;w_{20},0)$ has a phase that varies linearly with x' and describes the arc of a circle traced from the origin in the complex plane, as shown in Fig.1(b). It can be appreciated that when the pupil function is even, the MTF decreases rapidly with added defocus and, for $w_{20}\geq 1/2$ nulls and phase reversals are introduced.

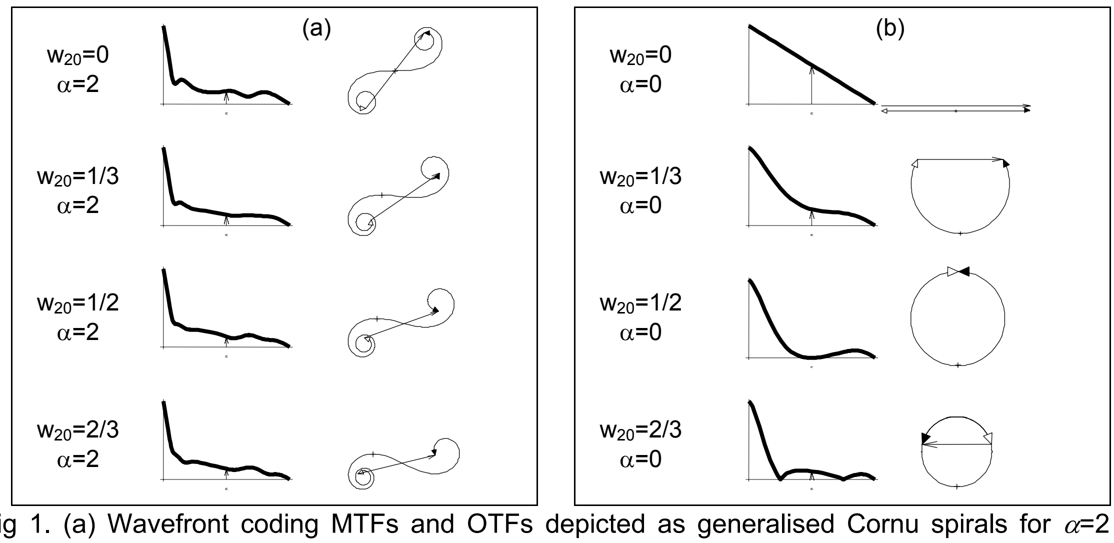

Fig 1. (a) Wavefront coding MTFs and OTFs depicted as generalised Cornu spirals for α=2 and ν=0.5. (b) Defocus MTFs and OTFs depicted as arc circles for α=0 and ν=0.5. Note the absence of nulls and phase reversal for the wavefront coding OTFs in contrast with the defocused OTFs.

We can use the geometry of the GCS both to understand the wavefront coding technique and to estimate important parameters such as an approximation to the value of the mean MTF, the maximum value of $|w_{20}|$ for which the MTF can be considered to be approximately constant and the actual amplitude and phase modulation of the OTF within this region. Previous analyses make approximations such that these modulations are zero and hence cannot be estimated.Due to the space limitations, only final results are shown.

The distance between the spirals' foci is proportionate to the mean value of the MTF as both ends of the spiral oscillate around them for increasing defocus. Therefore, the mean value of the MTF can be calculated by taking the limit $x'\to\pm\infty$ in Eq. (2), yielding

$$\left|\mathrm{L}\left(\nu;\alpha\right)\right| = 1/4\sqrt{3\alpha\nu} \quad (3)$$

which is independent of the defocus parameter and gives a straightforward estimation of the height reduction of the MTF.

The wavefront coded MTF remains practically constant up to a maximum value of tolerable defocus. At this point, one end of the spiral is completely unwrapped and thus the MTF drops completely. This can be obtained easily from the curvature of the GCS, $\kappa(x',\nu,w_{20},\alpha)=16\pi\nu(w_{20}+3\alpha x')$. It can thus be appreciated that the maximum value of w_{20} for which the MTF is approximately constant is that for which either end of the curve passes through the point of inflection, or zero curvature, that is, for $\kappa=0$. This occurs when $x'=|1-\nu|=|w_{20}|/3\alpha$ yielding a maximum value for tolerable defocus of $|w_{20}|=3\alpha(1-\nu)$.

Finally, the amplitude modulation of the MTF, $\mathrm{M}(\nu,w_{20},\alpha)$ as the spiral unravels can be calculated on the same geometrical grounds from the modulation of the resultant phasor. It can be shown that

$$\mathrm{M}\left(\nu;w_{20},\alpha\right) \leq 1/16\pi\nu\left(w_{20}-3\alpha\left(1-\nu\right)\right) \text{ for } 0<w_{20}<3\alpha \quad (4)$$

Conclusions

A novel approach based on geometrical analysis of plane curves has been shown to provide by simple means an evaluation method of wavefront coding performance and a unique insight into its physical principles. The curves, known as generalised Cornu spirals, were contructed from decomposition of the OTF into its fundamental Young's fringes phasors. Important parameters of the performance of a 1D wavefront coded system such as MTF suppression and modulation and maximum tolerable defocus were readily obtained.

Reference

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